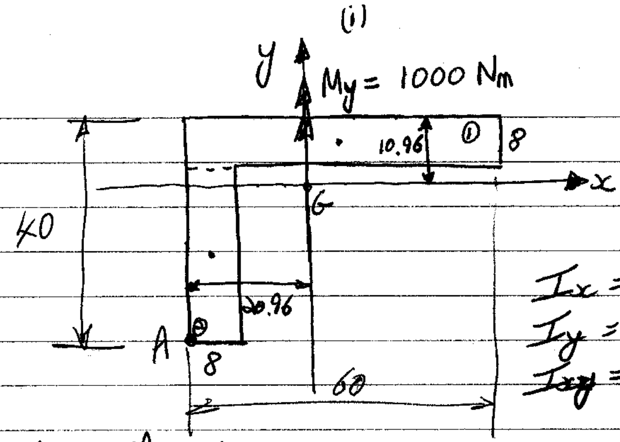
Q1

(i)



$I_x = 91,188.$ [redacted] mm^4
$I_y = 258,227.\ mm^4$
$I_{xy} = 86,818\ mm^4$

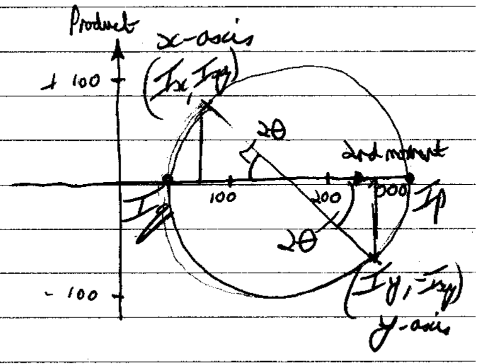
Principal 2nd Moments

Mohr's Circle Centre $= \frac{I_x + I_y}{2} = 174,708$

Radius $= \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} = 120,469$

$\therefore\ I_p = C + R = 295,177$
$I_q = C - R = 54,239$

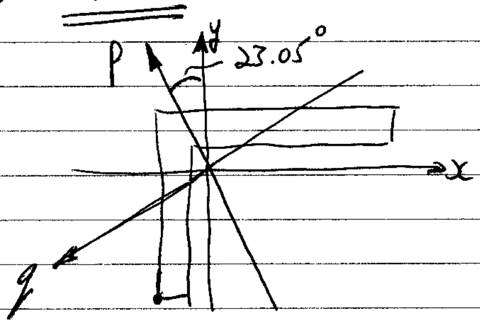
Angle of principal axes



$\sin 2\theta = \frac{I_{xy}}{R} = \frac{86,818}{120,469}$

$\therefore\ 2\theta = 46.1^\circ$
$\theta = 23.05^\circ$

Sketch



(ii)

Q1(cont) Co-ordinate Transformation equations

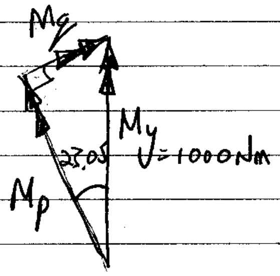
$$p = x\cos\theta + y\sin\theta$$
$$q = -x\sin\theta + y\cos\theta$$

$\theta = 90 + 23.05 = \underline{113.05^\circ}$

$\sin\theta = 0.92$
$\cos\theta = -0.392$

$$\therefore \quad p = -0.392x + 0.92y$$
$$q = -0.92x - 0.392y$$

Resolving Moment onto p & q axes



$$M_p = M_y \cos 23.05$$
$$= \underline{920\ Nm} \quad (+ve)$$

$$M_q = -M_y \sin 23.05$$
$$= \underline{-392\ Nm} \quad (-ve)$$

Co-ordinates of A

	x	y	p	q
Point A	−20.96	−29.04	−18.5	30.67

Bending Stress

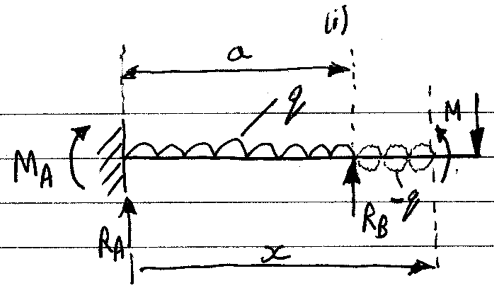
$$\sigma_{bA} = \frac{M_p q}{I_p} - \frac{M_q p}{I_q}$$

$$= \frac{920.10^3 . 30.67}{295,177} - \frac{-392.10^3 \times -18.5}{54,239}$$

$$= 95.59 - 133.70$$

$$= \underline{\underline{-38.11\ MPa}}$$

Q2 (i)



$P = 10kN$
$q = 5 kN/m$
$a = 1m$
$L = 1.5m$
$EI = 10^4 Nm^2$

Consider section at position x, & counterbalance UDL,

$$M - M_A - R_A x - R_B\langle x-a\rangle + \frac{qx^2}{2} - \frac{q\langle x-a\rangle^2}{2} = 0$$

$$EI\frac{d^2y}{dx^2} = -M = -M_A - R_A x - R_B\langle x-a\rangle + \frac{qx^2}{2} - \frac{q\langle x-a\rangle^2}{2}$$

$$EI\frac{dy}{dx} = -M_A x - \frac{R_A x^2}{2} - \frac{R_B\langle x-a\rangle^2}{2} + \frac{qx^3}{6} - \frac{q\langle x-a\rangle^3}{6} + A$$

when $x=0$ $\frac{dy}{dx}=0$ $\therefore$ $A=0$

$$EIy = -\frac{M_A x^2}{2} - \frac{R_A x^3}{6} - \frac{R_B\langle x-a\rangle^3}{6} + \frac{qx^4}{24} - \frac{q\langle x-a\rangle^4}{24} + B$$

when $x=0$ $y=0$ $\therefore$ $B=0$

When $x=a$ $y=0$

$$\therefore \quad 0 = -\frac{M_A a^2}{2} - \frac{R_A a^3}{6} + \frac{qa^4}{24}$$

$$\therefore \quad -12M_A a^2 - 4R_A a^3 + qa^4 = 0$$

or $\quad -12M_A - 4R_A a + qa^2 = 0 \qquad (1)$

Equilibrium of the whole beam

$\uparrow \quad R_A + R_B = P + qa \qquad (2)$

(ii)

Q2 (cont)

(A) $M_A + PL + qa.\frac{a}{2} - R_B a = 0$

$2M_A - 2R_B a + 2PL + qa^2 = 0$ (3)

Eliminate R_B from (2) & (3) to give,

$2M_A + 2PL + qa^2 - 2a(P + qa - R_A) = 0$

$\therefore 2M_A + 2PL + qa^2 - 2Pa - 2qa^2 + 2R_A a = 0$

$2M_A + 2P(L-a) + 2R_A a - qa^2 = 0$ (4)

Eliminating M_A from (1) & (4) gives,

$-4R_A a + qa^2 + 12P(L-a) + 12R_A a - 6qa^2 = 0$

$8R_A a - 5qa^2 + 12P(L-a) = 0$

$\therefore R_A = \frac{5qa^2 - 12P(L-a)}{8a}$

Substituting in values $R_A = \frac{5.5000.1^2 - 12.10,000(0.5)^2}{8.1} = -4375\ N$

$R_B = P + qa - R_A = 10,000 + 5000.1 + 4375 = 19,375\ N$

& $M_A = \frac{qa^2}{12} - \frac{R_A a}{3} = \frac{5000.1^2}{12} + \frac{4375.1}{3} = 1875\ Nm$

Deflection

when $x = 1.5$

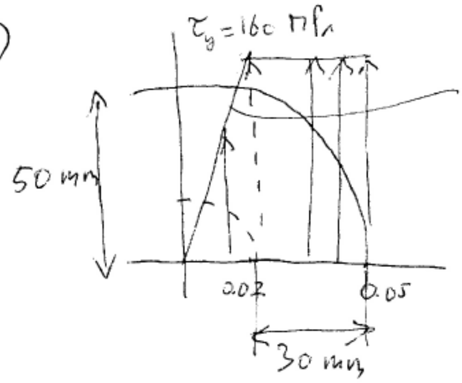
$EIy = -M_A\frac{1.5^2}{2} - R_A\frac{(1.5)^3}{6} - R_B\frac{(0.5)^3}{6} + \frac{5000(1.5)^4}{24} - \frac{5000(0.5)^4}{24}$

$EIy = -2109 + 2461 - 404 + 1055 - 13$
$= 990$

$\therefore y = 990/10^4 = 99\ mm$

Q3

(a)



slope $= \frac{160 \text{ MPa}}{0.02 \text{ m}} = 8000 \times 10^6$ Pa/m (N/m³)

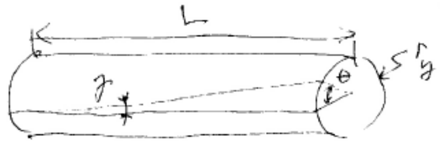
$$T = \int_A \tau r \, dA = T_{elastic} + T_{plastic}$$

$$T_{elastic} = \int_0^{0.02} (8000 \times 10^6 \, r) \times r \times 2\pi r \, dr = 16000\pi \times 10^6 \left.\frac{r^4}{4}\right]_0^{0.02} = 2.01 \times 10^3 \text{ Nm}$$

$$T_{plastic} = \int_{0.02}^{0.05} (160 \times 10^6) \times r \times 2\pi r \, dr = 320\pi \times 10^6 \left.\frac{r^3}{3}\right]_{0.02}^{0.05} = 39.2 \times 10^3 \text{ Nm}$$

$$\therefore T = (2.01 + 39.2) \times 10^3 \text{ Nm} = \underline{41.3 \text{ kNm}}$$

(b)



$r_y \theta = \gamma_y \times L$; $\gamma_y = \frac{\tau_y}{G} = \frac{160 \text{ MPa}}{80 \text{ GPa}}$ rad $= 2 \times 10^{-3}$ rad

$$\theta = \frac{\gamma_y \times L}{r_y}$$

$$= \frac{2 \times 10^{-3} \times 10 \text{ m}}{20 \times 10^{-3} \text{ m}} \text{ rad} = 1 \text{ rad} \quad (= \underline{57.3} \text{ deg})$$

(c) unloading:

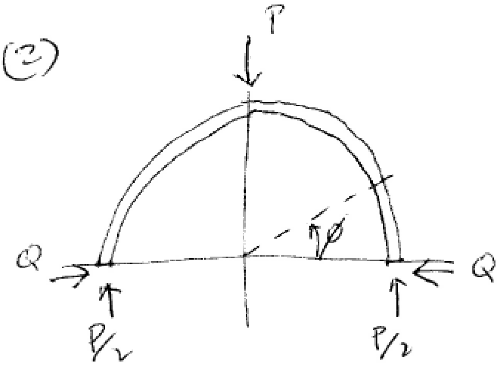
$\theta_{elastic} = \frac{T \times L}{J \times G}$; $J = \frac{\pi}{2} \times 0.05^4 = 9.8175 \times 10^{-6}$ m⁴

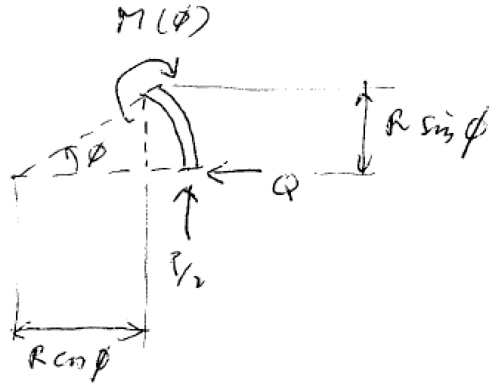
$$= \frac{41.3 \times 10^3 \times 10}{9.8175 \times 10^{-6} \times 80 \times 10^9} \text{ rad} = 0.525 \text{ rad} \ (= 30.1 \text{ deg}).$$

$\therefore \theta_{permanent\ twist} = (57.3 - 30.1)$ deg $= \underline{27.2}$ deg

Q4

(c)





Bending moment:

$$M(\phi) = \frac{P}{2}(R - R\cos\phi) - QR\sin\phi$$

Strain energy:

$$U = \frac{1}{2EI}\int_0^{\pi} M^2(\phi)\, R\, d\phi = \frac{1}{EI}\int_0^{\pi/2} M^2 R\, d\phi$$

No horizontal deflection at the end point:

$$\frac{\partial U}{\partial Q} = 0$$

$$\therefore \frac{1}{EI}\frac{\partial}{\partial Q}\int_0^{\pi/2} M^2 R\, d\phi = 0$$

$$\therefore 2\int_0^{\pi/2} M \times \left(\frac{\partial M}{\partial Q}\right) \times R\, d\phi = 0; \quad \text{where } \frac{\partial M}{\partial Q} = -R\sin\phi$$

$$\therefore \int_0^{\pi/2}\left[\frac{P}{2}(R - R\cos\phi) - QR\sin\phi\right](-R\sin\phi) \times R\, d\phi = 0$$

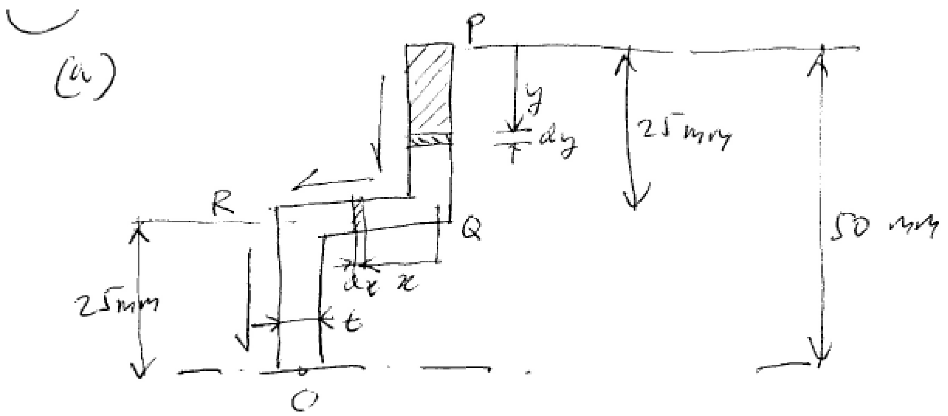
$$-\frac{R^3 P}{2}\underbrace{\int_0^{\pi/2}(1-\cos\phi)\sin\phi\, d\phi}_{\frac{1}{2}} + QR^3\underbrace{\int_0^{\pi/2}\sin^2\phi\, d\phi}_{\pi/4} = 0$$

($\sin\phi\, d\phi \rightarrow -\cos\phi$)

$$\therefore Q = \frac{P}{\pi}$$

Q5

(a)



$$\tau_{PQ}(y) = \frac{S}{It}\int y\,dA \; ; \quad \int y\,dA = \left(50 - \frac{y}{2}\right) \times ty$$

$$= \frac{S}{It} \times \left(50 - \frac{y}{2}\right) \times ty$$

$$\rightarrow S_{PQ} = \int_{y=0}^{y=25} \tau_{PQ}\, t\,dy = \frac{St}{I}\int_0^{25}\left(50 - \frac{y}{2}\right) y\,dy = 1.3\times10^4\,\frac{St}{I}$$

$$\tau_{QR}(x) = \frac{S}{It} \times \left[\left\{25t \times \left(25 + \frac{25}{2}\right)\right\} + \left\{xt \times 25\right\}\right]$$

$$= \frac{S}{It} \times \left[937.5\,t + 25x\,t\right]$$

$$\rightarrow S_{QR} = \int_{x=0}^{x=25} \tau_{QR}\, t\,dx$$

$$= \frac{St}{I}\int_0^{25}(937.5 + 25x)\,dx = 3.125\times10^4\,\frac{St}{I}$$

Torsion moment at point O :

$$M_O = S_{PQ} \times 25 \times 2 - S_{QR} \times 25 \times 2 = -S \times e$$

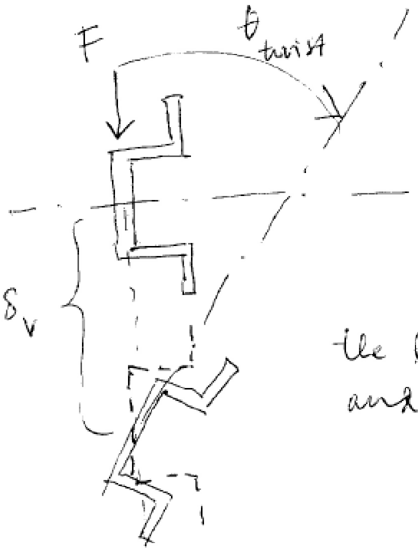
$$\therefore e = \frac{50 \times (S_{PQ} - S_{QR})}{-S} = 91.25\times10^4\,\frac{t}{I}$$

$$I = \frac{1}{12}(t)(100)^2 + 2\times\left[25t \times 25^2\right] = 11.46\times10^4\,t \text{ mm}^4$$

$$\therefore e = \frac{91.25\times10^4}{11.46\times10^4} = \underline{7.96} \text{ mm}$$

Q5 (cont)

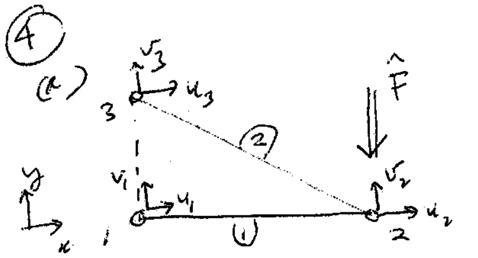
(b) The case where the external load is applied on the web.



the beam will be deflected vertically and twisted clock-wise.

Q6

④ (a)



i	A_i	L_i	E_i	θ_i
1	A	$L\sqrt{3}$	E	0°
2	A	2L	E	~~120°~~ 150°

$$k_e^{(i)} = \frac{A_i E_i}{L_i}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

$c = \cos\theta_i$
$s = \sin\theta_i$.

$$k_e^{(1)} = \frac{AE}{L\sqrt{3}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$k_e^{(2)} = \frac{AE}{2L}\begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

stiffness matrix ↘

Columns: u_1, v_1, u_2, v_2, u_3, v_3

$$\frac{AE}{L}\times\begin{bmatrix} \frac{1}{\sqrt{3}} & & -\frac{1}{\sqrt{3}} & & & \\ & & & & & \\ -\frac{1}{\sqrt{3}} & & \frac{1}{\sqrt{3}}+\frac{3}{8} & -\frac{\sqrt{3}}{8} & -\frac{3}{8} & +\frac{\sqrt{3}}{8} \\ & & -\frac{\sqrt{3}}{8} & \frac{1}{8} & \frac{\sqrt{3}}{8} & -\frac{1}{8} \\ & & -\frac{3}{8} & \frac{\sqrt{3}}{8} & \frac{3}{8} & -\frac{\sqrt{3}}{8} \\ & & +\frac{\sqrt{3}}{8} & -\frac{1}{8} & -\frac{\sqrt{3}}{8} & \frac{1}{8} \end{bmatrix}\begin{Bmatrix} u_1 = 0 \\ v_1 = 0 \\ u_2 \\ v_2 \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} = \begin{Bmatrix} F_1^x \\ F_1^y \\ F_2^x \\ F_2^y \\ F_3^x \\ F_3^y \end{Bmatrix}$$

F_2^x ← zero
$F_2^y = -\hat{F}$ ← magnitude?

(b)

$$\frac{AE}{L}\begin{bmatrix} \frac{1}{\sqrt{3}}+\frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & \frac{1}{8} \end{bmatrix}\times\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}\hat{F}$$

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \frac{L}{AE}\begin{bmatrix} \frac{1}{\sqrt{3}}+\frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{\sqrt{3}}{8} & \frac{1}{8} \end{bmatrix}^{-1}\times\begin{Bmatrix} 0 \\ -1 \end{Bmatrix}\hat{F} = \begin{Bmatrix} -3 \\ -13.1962 \end{Bmatrix}\times\frac{\hat{F}L}{AE}$$

-3 ← horizontal displacement
-13.1962 ← vertical displacement

Q6 (cont)

4) Forces

(e) ~~Reactions~~ (on the structure)

$F_1^x = -\frac{AE/L}{\sqrt{3}} \times u_2 + 0 \times v_2 = \frac{3}{\sqrt{3}} \hat{F} = \sqrt{3} F \; (\rightarrow)$

$F_1^y = 0 \times u_2 + 0 \times v_2 = 0$

$F_3^x = -\frac{3}{8} \frac{AE}{L} \times u_2 + \frac{\sqrt{3}}{8} \frac{AE}{L} \times v_2 = -1.7321 \hat{F}$
$= -\sqrt{3} \hat{F} \; (\leftarrow)$

$F_3^y = +\frac{\sqrt{3}}{8} \frac{AE}{L} \times u_2 - \frac{1}{8} \frac{AE}{L} \times v_2 = 1 \hat{F} \; (\uparrow)$

Reactions on the supports are equal and opposite in direction.

$R_1^x = -\sqrt{3} F \quad (\leftarrow)$

$R_1^y = 0$

$R_3^x = 1.732 F \; (\rightarrow)$

$R_3^y = -F \quad (\downarrow)$

} you can also combine these two to get the resultant magnitude and angle!

Q7

a)
The fatigue failure mechanism for an initially un-cracked component with a smooth (polished) surface can be split into three parts, namely crack initiation, crack propagation and final fracture, as follows: (i) Stage I crack growth: The micro-structural phenomenon which causes the initiation of a fatigue crack is the development of persistent slip bands at the surfaces of the specimen. These persistent slip bands are the result of dislocations moving along crystallographic planes leading to both slip band intrusions and extrusions on the surface. These act as excellent stress concentrations and can thus lead to crack initiation. Crystallographic slip is primarily controlled by shear stresses rather than normal stresses so that fatigue cracks initially tend to grow in a plane of maximum shear stress range. This stage leads to short cracks, usually only of the order o0f a few grains. Stage II crack growth: As cycling continues, the fatigue cracks tend to coalesce and grow along planes of maximum tensile stress range. (iii) Final fracture; this occurs when the crack reaches a critical length and results in either ductile tearing (plastic collapse) at one extreme, or cleavage (brittle fracture) at the other extreme.

b)
Having a larger section at the hole area or design with stress relief groves.

c)
First find the critical crack length, a_c for σ_{max}

Use $K_{IC} = Y\sigma\sqrt{\pi a}$

$$K_{IC} = 1.12\sigma\sqrt{\pi a}$$

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{1.12\sigma_{fail}}\right)^2 = \frac{1}{\pi}\left(\frac{104\times10^6}{1.12\times 200\times10^6}\right)^2 = 0.06m$$

From the question we know that $\Delta\sigma$=200MPa.

Using $N_f = \frac{1}{AY^2(\Delta\sigma)^2\pi}\ln\left(\frac{a_c}{a_i}\right)$, we can find that the fatigue life

$N_f = \frac{1}{6.9\times10^{-30}\times(1.12)^2\times(200\times10^6)^2\pi}\ln\left(\frac{69\times10^3}{0.5\times10^3}\right)$ =4.53013x10^{12}cycles to failure.

Q8

1. after assembly, the radial interference pressure, *p*, will be the same on both cylinders, i.e. Cylinder 1 will have an external pressure, *p*, and Cylinder 2 will have an internal pressure, *p*
2. The decrease in the outside radius of Cylinder 1, i_1, plus the increase in the inside radius of Cylinder 2, i_2, will be equal to the radial interference, i.e. $i = i_1 + i_2$

For cylinder (1):

$$\sigma_r = A_1 - \frac{B_1}{r^2}$$

and $\sigma_\theta = A_1 + \frac{B_1}{r^2}$

at r = 20mm, σ_r = 0,

$$\therefore B_1 = 400A_1$$

at r = 40mm (no significant difference with 40.05mm), σ_r = -*p*

$$\therefore -p = A_1 - \frac{20^2}{40^2}A_1 = A_1 - \frac{400}{1600}A_1$$

$$ie\ A_1 = -\frac{4}{3}p$$

$$B_1 = -\frac{1600}{3}p$$

Thus, $(\sigma_r)_1 = -\frac{4p}{3}\left(1 - \frac{400}{r^2}\right)$

and $(\sigma_\theta)_1 = -\frac{4p}{3}\left(1 + \frac{400}{r^2}\right)$

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right) = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

At the outside of cylinder (1), r = 40mm,

Q8 (cont)

$$\frac{-i_1}{40} = \frac{1}{200{,}000}(\sigma_\theta - \nu\sigma_r)$$

$$ie\ \frac{-i_1}{40} = \frac{1}{200{,}000}\left(-\frac{4p}{3}\right)\left(1+\frac{400}{1600} - \nu\left(1-\frac{400}{1600}\right)\right)$$

$$i_1 = \frac{8p}{30000}\left(\frac{5}{4} - \frac{3\nu}{4}\right) = \frac{2p}{30000}(5-3\nu)$$

For cylinder (2):

$$\sigma_r = A_2 - \frac{B_2}{r^2}$$

and $\sigma_\theta = A_2 + \frac{B_2}{r^2}$

At r = 60mm, σ_r = 0

$$\therefore B_2 = 3600A_2$$

At r = 40mm, σ_r = $-p$

$$\therefore -p = A_2 - \frac{60^2}{40^2}A_2 = A_2 - \frac{3600}{1600}A_2$$

$$ie\ A_2 = \frac{4}{5}p$$

and

$$B_2 = 3600 \times \frac{4}{5}p$$

Thus, $(\sigma_r)_2 = \frac{4p}{5}\left(1 - \frac{3600}{r^2}\right)$

and $(\sigma_\theta)_1 = -\frac{4p}{3}\left(1 + \frac{400}{r^2}\right)$

Q8 (cont)

$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}\left(\sigma_\theta - \nu(\sigma_r + \sigma_z)\right) = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

At the inside of cylinder (2), *r* = 40mm,

$$\frac{+i_2}{40} = \frac{1}{200{,}000}\left(\frac{4p}{5}\right)\left(1 + \frac{3600}{1600} - \nu\left(1 - \frac{3600}{1600}\right)\right)$$

$$ie\ i_2 = \frac{8p}{50000}\left(\frac{13}{4} + \frac{5\nu}{4}\right)$$

$$\therefore i_2 = \frac{2p}{50000}\left(13 + 5\nu\right)$$

But $i_1 + i_2 = i = 0.05$mm

$$\therefore \frac{2p}{30000}\left(5 - 3\nu\right) + \frac{2p}{50000}\left(13 + 5\nu\right) = 0.05$$

$$\frac{10p}{30000} - \frac{2\nu p}{10000} + \frac{26p}{50000} + \frac{2\nu p}{10000} = 0.05$$

$$\frac{50p + 78p}{150{,}000} = 0.05$$

$$ie\ p = \frac{7500}{128}\,\text{N/mm}^2 = 58.6\,\text{N/mm}^2$$

For cylinder (1),

$$\left(\sigma_r\right)_1 = -78.1\left(1 - \frac{400}{r^2}\right)$$

$$\left(\sigma_\theta\right)_1 = -78.1\left(1 + \frac{400}{r^2}\right)$$

and for cylinder (2),

$$\left(\sigma_r\right)_2 = 46.9\left(1 - \frac{3600}{r^2}\right)$$

Q8 (cont)

$$(\sigma_\theta)_2 = 46.9\left(1+\frac{3600}{r^2}\right)$$